

THE BOUNDEDNESS OF OPERATORS IN MUCKENHOUT WEIGHTED MORREY SPACES VIA EXTRAPOLATION TECHNIQUES AND DUALITY

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ABSTRACT. The bidual of the closure of smooth functions with respect to the Morrey norm coincides with the Morrey space. This assertion is generalized to some Muckenhoupt weighted Morrey spaces. We combine this fact with basic extrapolation techniques due to Rubio de Francia adapted to weighted Morrey spaces. This leads to new results on the boundedness of operators even for the unweighted case.

1. INTRODUCTION

An important tool of modern harmonic analysis is the extrapolation theorem due to Rubio de Francia. For a given operator T we suppose that for some p_0 , $1 \leq p_0 < \infty$, and for every weight belonging to the Muckenhoupt class A_{p_0} the inequality

$$(1) \quad \|Tf\|_{L_{p_0,w}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x) dx \right)^{\frac{1}{p_0}} \leq c \|f\|_{L_{p_0,w}(\mathbb{R}^n)}$$

holds, where the constant c is independent of f but can depend on $[w]_{A_{p_0}}$. Then for every p , $1 < p < \infty$, and every $w \in A_p$ there exists a constant depending on $[w]_{A_p}$ such that

$$(2) \quad \|Tf\|_{L_{p,w}(\mathbb{R}^n)} \leq c \|f\|_{L_{p,w}(\mathbb{R}^n)}$$

(cf. [Rub82, Rub84, CMP11, Duo13]). In this paper we consider Muckenhoupt weighted Morrey spaces $L_p^r(w, \mathbb{R}^n)$ collecting all locally integrable complex-valued functions given on \mathbb{R}^n with

$$\|f\|_{L_p^r(w, \mathbb{R}^n)} = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{r}{n})} \left(\int_{Q_{J,M}} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty$$

where $Q_{J,M} = 2^{-J}(M + [-1, 1]^n)$, $1 < p < \infty$ and $-\frac{n}{p} \leq r < 0$. The spaces $L_p^r(w, \mathbb{R}^n)$ coincide with the unweighted Morrey spaces $L_p^r(\mathbb{R}^n)$ if $w(x) = 1$ and furthermore with $L_{p,w}(\mathbb{R}^n)$ if $r = -\frac{n}{p}$. It is our aim to show that (1) also implies

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that for every p , $1 < p < \infty$, every r , $-\frac{n}{p} \leq r < 0$, and every $w \in A_p$ there exists a constant depending on $[w]_{A_p}$ such that

$$(3) \quad \|Tf\|_{L_p^r(w, \mathbb{R}^n)} \leq c \|f\|_{L_p^r(w, \mathbb{R}^n)}$$

for $D(\mathbb{R}^n)$. If we assume in addition that T admits a unique and continuous extension from $D(\mathbb{R}^n)$ on $\mathring{L}_p^r(w, \mathbb{R}^n)$ where $\mathring{L}_p^r(w, \mathbb{R}^n)$ stands for the completion of $D(\mathbb{R}^n)$ with respect to $\|\cdot\|_{L_p^r(w, \mathbb{R}^n)}$, then we achieve

$$(4) \quad T : \mathring{L}_p^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n).$$

Note that linearity of T is sufficient for the existence of this unique and continuous extension but also other operators as the maximal operator or maximally truncated singular integral operators are admissible (cf. (65) for another sufficient condition on T). This is a new method to prove (4) even in the unweighted Morrey spaces $L_p^r(\mathbb{R}^n)$. In many previous papers inequalities of type (4) are deduced assuming the estimate

$$(5) \quad |(Tf)(y)| \leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{|y-x|^n} dx \quad \text{for all } f \in D(\mathbb{R}^n) \text{ and } y \notin \text{supp}(f)$$

and using the (weighted) $L_{p_0}(\mathbb{R}^n)$ boundedness of T (cf. [Nak94, DYZ98, GAKS11, Gul12, Mus12, RS14, PT15, Wan16] and the references given there or similar size estimates of the operator cf. [Alv96, SFZ13]). Sharpening (4) to

$$(6) \quad T : \mathring{L}_p^r(w, \mathbb{R}^n) \hookrightarrow \mathring{L}_p^r(w, \mathbb{R}^n)$$

we find extensions of T via (bi-)duality on $L_p^r(w, \mathbb{R}^n)$ such that

$$(7) \quad T : L_p^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n).$$

This is based on the duality result

$$(8) \quad (\mathring{L}_p^r(w, \mathbb{R}^n))' \cong L_p^r(w, \mathbb{R}^n)$$

which extends the corresponding unweighted duality assertion observed in [AX12] and proved completely in [RT14]. Our method proving (4) and (7) using (1) instead of (5) leads to new results on the boundednesses of operators even in the unweighted Morrey spaces $L_p^r(\mathbb{R}^n)$. As examples Hörmander-Mikhlin type multipliers, Marcinkiewicz multipliers and commutators will be considered. Let us mention that in many related papers (starting from Peetre [Pee66] and many following scholars) dealing with various generalizations of $L_p^r(\mathbb{R}^n)$ the non-density of $D(\mathbb{R}^n)$ in $L_p^r(\mathbb{R}^n)$ is not taken into account. Then the use of (5) for all $f \in L_p^r(\mathbb{R}^n)$ (instead of $f \in D(\mathbb{R}^n)$) has to be justified, in particular, for singular integrals and multipliers. Moreover, one has also to clarify in which way one extends the operator T given on some $L_{p_0}(\mathbb{R}^n)$ space (or given on $D(\mathbb{R}^n)$) to $L_p^r(\mathbb{R}^n)$ (whenever necessary cf. singular integrals and multipliers). For a detailed discussion we refer to the forerunner results [Alv96, AX12, RT13, Ros13, RT14, Tri14, RS14, Ad15]. Based on (6) and (7) we are able to complete previous results for singular integrals (Calderón-Zygmund operators), multipliers, commutators, ... with respect to (4). In contrast to [Alv96, AX12, Ad15] we are working on $\mathring{L}_p^r(w, \mathbb{R}^n)$ (instead on $\mathring{L}_p^r(w, \mathbb{R}^n)'$). [AX12, Ad15] use also (1) to obtain partial results in the unweighted situation $L_p^r(\mathbb{R}^n)$. By the fact that there are also negative results with respect to

interpolation of Morrey spaces (cf. [BRV99]) it was not clear that extrapolation will work.

After introducing the notation and some preliminaries in Section 2, Section 3 is concerned with preparations which are needed to prove the duality result (8) in Section 4. Here we investigate basic embeddings, density and separability of weighted (pre-)dual Morrey spaces. Using duality in Morrey spaces, (8), we prove (3) in Section 5 generalizing ideas of [CMP11, Theorem 4.6] and [CGCMP06]. In Section 6 we present the main result of the paper which states that (1) and the existence of an unique and continuous extension of T from $D(\mathbb{R}^n)$ to $\mathring{L}_p^r(w, \mathbb{R}^n)$ imply (6) and (7) (cf. Theorem 6.1). Moreover, if T is formally self-adjoint in $L_{p_0, w}(\mathbb{R}^n)$, then we have also

$$T : \mathring{L}_p^r(w, \mathbb{R}^n)' \hookrightarrow \mathring{L}_p^r(w, \mathbb{R}^n)'$$

where $\mathring{L}_p^r(w, \mathbb{R}^n)'$ admits an atomic characterization (cf. Definition 3.4 and Theorem 4.8). We shall apply our general results to distinguished operators (Calderón-Zygmund operators, Hörmander-Mikhlin type multipliers, Marcinkiewicz multipliers, maximal Carleson operator, commutators). Finally, we explain how one can lift these results to the vector-valued case. In the last section we characterize the associated spaces of the Morrey spaces.

2. NOTATION, MORREY SPACES AND PRELIMINARIES

We use standard notation. Let \mathbb{N} be the collection of all natural numbers and \mathbb{R}^n be the Euclidean n -space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \mathbb{R}^n . Let $D(\mathbb{R}^n)$ be the collection of all infinitely differentiable functions with compact support in \mathbb{R}^n , where the support of a function f is abbreviated by $\text{supp}(f)$. Furthermore, $L_{p, w}(\mathbb{R}^n)$ with $1 \leq p < \infty$ is the complex Banach space of functions whose p -th power is integrable with respect to the weight $w : \mathbb{R}^n \rightarrow [0, \infty]$ and which is normed by

$$\|f\|_{L_{p, w}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

Moreover, we write $w(M) = \int_M w(x) dx$ for the measurable subset M of \mathbb{R}^n . We similarly define $L_{p, w}(M)$. If $w(x) = 1$, we simply write $L_p(M)$, $\|\cdot\|_{L_p(M)}$ and $|M|$. Furthermore, χ_M denotes the characteristic function of M . As usual, \mathbb{Z} is the collection of all integers; and \mathbb{Z}^n where $n \in \mathbb{N}$ denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Moreover, $L_1^{\text{loc}}(\mathbb{R}^n)$ collects all equivalence classes of almost everywhere coinciding measurable complex locally integrable functions, hence $f \in L_1(M)$ for any bounded measurable set M of \mathbb{R}^n . If Q denotes a cube in \mathbb{R}^n (whose sides are parallel to the coordinate axes), then dQ stands for the concentric cube with side-length $d > 1$ times of the side-length of Q . For any $p \in (1, \infty)$ we denote by p' the conjugate index, namely, $1/p + 1/p' = 1$. For Banach spaces X and Y we denote by

$$T : X \hookrightarrow Y$$

a bounded operator mapping X into Y . That is, we have

$$\|Tx|Y\| \leq c \|x|X\|$$

for all $x \in X$ where the constant c is independent of x . The concrete value of constants may vary from one formula to the next, but remains the same within one chain of (in)equalities. Finally, $A \cong B$ means that there are two constants $c, C > 0$ such that $cA \leq B \leq CA$.

Definition 2.1. We say that $w \in L_1^{\text{loc}}(\mathbb{R}^n)$ with $w > 0$ almost everywhere belongs to A_p for $1 < p < \infty$ if

$$(9) \quad [w]_{A_p} \equiv \sup_Q \frac{w(Q)}{|Q|} \left(\frac{w^{1-p'}(Q)}{|Q|} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q in \mathbb{R}^n (whose sides are parallel to the coordinate axes). The value $[w]_{A_p}$ is called the A_p constant of the *Muckenhoupt weight* w .

We define some Muckenhoupt weighted Morrey spaces.

Definition 2.2. For $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$ and $w \in A_p$ we define

$$L_p^r(w, \mathbb{R}^n) \equiv \{f \in L_1^{\text{loc}}(\mathbb{R}^n) : \|f\|_{L_p^r(w, \mathbb{R}^n)} < \infty\}$$

with the norm

$$\|f\|_{L_p^r(w, \mathbb{R}^n)} \equiv \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|f\|_{L_{p,w}(Q_{J,M})}.$$

Recall that $Q_{J,M} = 2^{-J}(M + [-1, 1]^n)$, $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$ is a dyadic cube with side length 2^{-J+1} centered at $2^{-J}M$.

Remark 2.3. We observe that $L_p^{-n/p}(w, \mathbb{R}^n) = L_{p,w}(\mathbb{R}^n)$. If $w(x)dx = dx$, $L_p^r(w, \mathbb{R}^n)$ coincides with the unweighted Morrey spaces $L_p^r(\mathbb{R}^n)$. Furthermore, we mention that the a priori assumption $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ can be omitted. Indeed, if f is a measurable complex function defined on \mathbb{R}^n , then $\|f\|_{L_p^r(w, \mathbb{R}^n)} < \infty$, Hölder's inequality and (9) yield

$$\begin{aligned} \int_{Q_{J,M}} |f(x)| dx &\leq \|f\|_{L_{p,w}(Q_{J,M})} w^{1-p'}(Q_{J,M})^{\frac{1}{p'}} \\ &\leq [w]_{A_p}^{\frac{1}{p}} \|f\|_{L_{p,w}(Q_{J,M})} \frac{|Q_{J,M}|}{w(Q_{J,M})^{\frac{1}{p}}} \\ &\leq [w]_{A_p}^{\frac{1}{p}} \|f\|_{L_p^r(w, \mathbb{R}^n)} w(Q_{J,M})^{\frac{r}{n}} < \infty \end{aligned}$$

for all $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$. Hence, $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Remark 2.4. Let $1 < p < \infty$ and $w \in A_p$. Then for a cube Q in \mathbb{R}^n (whose sides are parallel to the coordinate axes) and a measurable subset $S \subset Q$ we have

$$(10) \quad \frac{w(Q)}{w(S)} \leq c \left(\frac{|Q|}{|S|} \right)^p$$

where the constant c does not depend on Q and S cf. [Duo01, (7.3)]. Moreover, w satisfies the reverse doubling condition, i.e. there exists a constant $0 < c < 1$ such that

$$(11) \quad w(Q) \leq c w(2Q)$$

holds for arbitrary cubes Q in \mathbb{R}^n (whose sides are parallel to the coordinate axes) where the constant c does not depend on the center and side-length of the cubes Q cf. [Duo01, Lemma 7.5].

Lemma 2.5. *Let $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$ and $w \in A_p$. Then*

$$(12) \quad \|f\|_{L_p^r(w, \mathbb{R}^n)} \cong \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(\widetilde{Q_{J,M}})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(\widetilde{Q_{J,M}})}$$

$$(13) \quad \cong \sup_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q(x, J))}$$

for all $f \in L_1^{loc}(\mathbb{R}^n)$ where $\widetilde{Q_{J,M}} = 2^{-J}(M + [0, 1]^n)$ and $Q(x, J) \equiv x + 2^{-J}[-1, 1]^n$.

Proof. We observe

$$\begin{aligned} & w(Q_{J,M})^{-(\frac{1}{p} + \frac{r}{n})p} \|f\|_{L_{p,w}(Q_{J,M})}^p \\ & \leq 2^n \sup_{\widetilde{M} \in \mathbb{Z}^n: \widetilde{Q_{J,\widetilde{M}}} \subset Q_{J,M}} w(\widetilde{Q_{J,\widetilde{M}}})^{-(\frac{1}{p} + \frac{r}{n})p} \|f\|_{L_{p,w}(\widetilde{Q_{J,\widetilde{M}}})}^p \end{aligned}$$

and hence

$$w(Q_{J,M})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q_{J,M})} \leq c \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(\widetilde{Q_{J,M}})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(\widetilde{Q_{J,M}})}$$

where the constant c does not depend on $J \in \mathbb{Z}$ and $M \in \mathbb{Z}^n$. Furthermore, $\widetilde{Q_{J,M}} \subset Q_{J,M}$ yields

$$\begin{aligned} & w(\widetilde{Q_{J,M}})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(\widetilde{Q_{J,M}})} \\ & \leq \left(\frac{w(Q_{J,M})}{w(\widetilde{Q_{J,M}})} \right)^{\frac{1}{p} + \frac{r}{n}} w(Q_{J,M})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q_{J,M})}. \end{aligned}$$

Since

$$\frac{w(Q_{J,M})}{w(\widetilde{Q_{J,M}})} \leq c$$

where the constant c does not depend on $J \in \mathbb{Z}$ and $M \in \mathbb{Z}^n$ (cf. (10)), we obtain the equivalence (12). Moreover,

$$\|f\|_{L_p^r(w, \mathbb{R}^n)} \leq \sup_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q(x, J))}$$

is obvious. Now, we choose a cube $Q(x, J)$. We fix then $\widetilde{M} \in \mathbb{Z}$ such that $\widetilde{M}2^{-J+1} \leq x_j < (\widetilde{M} + 1)2^{-J+1}$ where $x = (x_1, \dots, x_n)$ and define

$$(14) \quad M_j \equiv \begin{cases} \widetilde{M}, & x_j - \widetilde{M}2^{-J+1} \leq 2^{-J} \\ \widetilde{M} + 1, & (\widetilde{M} + 1)2^{-J+1} - x_j < 2^{-J} \end{cases}$$

for $j = 1, \dots, n$. Therefore, $Q(x, J) \subset Q_{J-1,M}$ where $M = (M_1, \dots, M_n)$ and

$$\begin{aligned} & w(Q(x, J))^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q(x, J))} \\ & \leq \left(\frac{w(Q_{J-1,M})}{w(Q(x, J))} \right)^{\frac{1}{p} + \frac{r}{n}} w(Q_{J-1,M})^{-(\frac{1}{p} + \frac{r}{n})} \|f\|_{L_{p,w}(Q_{J-1,M})}. \end{aligned}$$

Using (10) the equivalence (13) follows. \square

Remark 2.6. By means of (10) one can show analogously

$$(15) \quad \|f|L_p^r(w, \mathbb{R}^n)\| \cong \sup_{x \in \mathbb{R}^n, R > 0} w(B_R(x))^{-(\frac{1}{p} + \frac{r}{n})} \|f|L_{p,w}(B_R(x))\|$$

where $B_R(x)$ stands for the ball with radius R centered at $x \in \mathbb{R}^n$.

3. NON-SEPARABILITY, DENSITY AND EMBEDDINGS OF WEIGHTED MORREY SPACES

Proposition 3.1. Let $1 < p \leq \tilde{p} < \infty$, $-\frac{n}{p} \leq -\frac{n}{\tilde{p}} \leq r < 0$ and let $w \in A_p$. Then

$$D(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n) \hookrightarrow L_{u,w}(\mathbb{R}^n) \hookrightarrow L_{\tilde{p}}^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n)$$

where $u \equiv -\frac{n}{r}$.

Proof. Let $f \in L_{\tilde{p}}^r(w, \mathbb{R}^n)$ and $Q \equiv Q_{J,M}$. Hölder's inequality yields

$$\begin{aligned} w(Q)^{-(\frac{1}{p} + \frac{r}{n})} \|f|L_{p,w}(Q)\| &\leq w(Q)^{-\frac{1}{p} + \frac{r}{n}} \|f|L_{\tilde{p},w}(Q)\| w(Q)^{(1 - \frac{r}{p})\frac{1}{p}} \\ &\leq w(Q)^{-\frac{1}{p} + \frac{r}{n}} \|f|L_{\tilde{p},w}(Q)\| \leq \|f|L_p^r(w, \mathbb{R}^n)\| \end{aligned}$$

and hence $L_{\tilde{p}}^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n)$. Thus, observing $L_{u,w}(\mathbb{R}^n) = L_{-n/r,w}^r(\mathbb{R}^n)$ and $1 < \tilde{p} \leq -n/r$ we obtain the assertion. \square

The following proposition is a generalization of [RT14, Prop. 3.7] to weighted Morrey spaces.

Proposition 3.2. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $w \in A_p$ and let $u \equiv -\frac{n}{r}$. Then the spaces $L_p^r(w, \mathbb{R}^n)$ are non-separable. Furthermore, neither $D(\mathbb{R}^n)$ nor $S(\mathbb{R}^n)$ nor $L_{u,w}(\mathbb{R}^n)$ nor $L_{\infty}^{\text{comp}}(\mathbb{R}^n)$ is dense in $L_p^r(w, \mathbb{R}^n)$ where $L_{\infty}^{\text{comp}}(\mathbb{R}^n)$ denotes the collection of all compactly supported almost everywhere bounded functions.

Proof. Step 1. First we prove the non-separability. Let

$$(16) \quad Q_l \equiv 2^{-l}((2, \dots, 2) + [0, 1]^n), \quad l \in \mathbb{N}, l \geq 2,$$

be disjoint cubes with $Q_l \subset Q = [-1, 1]^n$ and let $2_n \equiv (2, \dots, 2)$. Let

$$(17) \quad f^\lambda \equiv \sum_{l=2}^{\infty} \lambda_l w(Q_l)^{\frac{r}{n}} \chi_l$$

where χ_l is the characteristic function of Q_l and

$$(18) \quad \lambda = \{\lambda_l\}_{l=2}^{\infty} \quad \text{with either } \lambda_l = 1 \text{ or } \lambda_l = -1.$$

Let $J \in \mathbb{Z}$ and $M \in \mathbb{Z}^n$. Taking into account the construction of $\{Q_l\}$ and $\widetilde{Q_{J,M}} = 2^{-J}(M + [0, 1]^n)$ we observe that there are just three cases:

- $\widetilde{Q_{J,M}}$ does not intersect another cube of $\{Q_l\}$,
- one of the cubes of $\{Q_l\}$ has nonempty intersection with $\widetilde{Q_{J,M}}$ (either $Q_{l_0} \subset \widetilde{Q_{J,M}}$ or $\widetilde{Q_{J,M}} \subset Q_{l_0}$ for some $l_0 \in \mathbb{N}$, $l_0 \geq 2$ and $\widetilde{Q_{J,M}} \cap Q_l = \emptyset$ for all $l \in \mathbb{N}$ with $l \neq l_0$)
- or $\widetilde{Q_{J,M}}$ does intersect infinitely many cubes of the family $\{Q_l\}_l$, in that case we have $\left\{\bigcup_{l:l \geq l_0} Q_l\right\} \subset \widetilde{Q_{J,M}}$ and $\widetilde{Q_{J,M}} \cap \left\{\bigcup_{l:l < l_0} Q_l\right\} = \emptyset$ for an appropriate $l_0 \in \mathbb{N}_0$ with $l_0 \geq 2$.

At first we treat the case $\widetilde{Q_{J,M}} \subset Q_{l_0}$ and $\widetilde{Q_{J,M}} \cap Q_l = \emptyset$ for $l \neq l_0$. For $\widetilde{Q_{J,M}} \subset Q_{l_0}$ we observe then

$$(19) \quad w(\widetilde{Q_{J,M}})^{-\left(\frac{1}{p} + \frac{r}{n}\right)p} \int_{\widetilde{Q_{J,M}}} |f^\lambda(x)|^p w(x) dx = \left(\frac{w(\widetilde{Q_{J,M}})}{w(Q_{l_0})} \right)^{-\frac{r}{n}p} \leq 1$$

by $r < 0$. In the case $Q_{l_0} \subset \widetilde{Q_{J,M}}$ and $\widetilde{Q_{J,M}} \cap Q_l = \emptyset$ for $l \neq l_0$ we obtain

$$(20) \quad w(\widetilde{Q_{J,M}})^{-\left(\frac{1}{p} + \frac{r}{n}\right)p} \int_{\widetilde{Q_{J,M}}} |f^\lambda(x)|^p w(x) dx = \left(\frac{w(Q_{l_0})}{w(\widetilde{Q_{J,M}})} \right)^{\left(\frac{1}{p} + \frac{r}{n}\right)p} \leq 1$$

by $\frac{1}{p} + \frac{r}{n} > 0$. It remains the case that $\left\{ \bigcup_{l:l \geq l_0} Q_l \right\} \subset \widetilde{Q_{J,M}}$ and $\widetilde{Q_{J,M}} \cap \left\{ \bigcup_{l:l < l_0} Q_l \right\} = \emptyset$ where $l_0 \in \mathbb{N}_0$ with $l_0 \geq 2$. Then $\widetilde{Q_{J,M}} = \widetilde{Q_{l_0-2,0_n}} = 2^{2-l_0}[0, 1]^n$ and

$$(21) \quad w(\widetilde{Q_{J,M}})^{-\left(\frac{1}{p} + \frac{r}{n}\right)p} \int_{\widetilde{Q_{J,M}}} |f^\lambda(x)|^p w(x) dx = \sum_{l:l \geq l_0} \left(\frac{w(Q_l)}{w(\widetilde{Q_{l_0-2,0_n}})} \right)^{\left(\frac{1}{p} + \frac{r}{n}\right)p} < \infty,$$

where the last inequality holds by the fact that w satisfies the reverse doubling condition. Indeed, the reverse doubling condition (11) yields

$$(22) \quad w(Q_{l,2_n}) < cw(2Q_{l,2_n})$$

for a constant $c < 1$ which does not depend on $l \in \mathbb{N}$, where we recall $Q_{l,2_n} = 2^{-l}((2, \dots, 2) + [-1, 1]^n)$. Thus,

$$(23) \quad \begin{aligned} & \sum_{l:l \geq l_0} \left(\frac{w(Q_l)}{w(\widetilde{Q_{l_0-2,0_n}})} \right)^{\left(\frac{1}{p} + \frac{r}{n}\right)p} \leq \sum_{l:l \geq l_0} \left(\frac{w(Q_{l,2_n})}{w(\widetilde{Q_{l_0-2,0_n}})} \right)^{\left(\frac{1}{p} + \frac{r}{n}\right)p} \\ & \leq \sum_{l:l \geq l_0} \left[c^{l-l_0} \left(\frac{w(2^{l-l_0}Q_{l,2_n})}{w(\widetilde{Q_{l_0-2,0_n}})} \right) \right]^{\left(\frac{1}{p} + \frac{r}{n}\right)p} \\ & \leq \left(\frac{w(Q_{l_0-2,0_n})}{w(\widetilde{Q_{l_0-2,0_n}})} \right)^{\left(\frac{1}{p} + \frac{r}{n}\right)p} \sum_{l:l \geq l_0} c^{(l-l_0)\left(\frac{1}{p} + \frac{r}{n}\right)p}, \end{aligned}$$

where the last inequality holds by

$$2^{l-l_0}Q_{l,2_n} = 2^{-l}2_n + 2^{-l_0}[-1, 1]^n \subset Q_{l_0-2,0_n} = 2^{2-l_0}[-1, 1]^n$$

and the right-hand side of (23) is finite by (10) and $c < 1$. Hence $f^\lambda \in L_p^r(w, \mathbb{R}^n)$ by (19)-(21) and Lemma 2.5. If $\lambda^1 \equiv \{\lambda_l^1\}_l$ and $\lambda^2 \equiv \{\lambda_l^2\}_l$ are two different admitted sequences, then one has $\lambda_{l_0}^1 = 1$ and $\lambda_{l_0}^2 = -1$ for some $l_0 \in \mathbb{N}$ with $l_0 \geq 2$ and moreover by Lemma 2.5

$$(24) \quad \begin{aligned} & \|f^{\lambda^1} - f^{\lambda^2}\|_{L_p^r(w, \mathbb{R}^n)} \\ & \geq c w(Q_{l_0})^{-\left(\frac{1}{p} + \frac{r}{n}\right)p} \left(\int_{Q_{l_0}} |(f^{\lambda^1} - f^{\lambda^2})(x)|^p w(x) dx \right)^{1/p} = 2c. \end{aligned}$$

But the set of all these admitted functions f^λ is non-countable, having the cardinality of \mathbb{R} . Then it follows from (24) that $L_p^r(w, \mathbb{R}^n)$ is not separable.

Step 2. The separable Lebesgue space $L_{u,w}(\mathbb{R}^n)$ is continuously embedded into the non-separable space $L_p^r(w, \mathbb{R}^n)$. This shows that this embedding cannot be dense. Moreover, $D(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ and $L_\infty^{\text{comp}}(\mathbb{R}^n)$ are subsets of $L_{u,w}(\mathbb{R}^n)$ and hence not dense in $L_p^r(w, \mathbb{R}^n)$. \square

Definition 3.3. Let $1 < p < \infty$, $-\frac{n}{p} \leq r < 0$ and let $w \in A_p$. Then $\mathring{L}_p^r(w, \mathbb{R}^n)$ is the completion of $D(\mathbb{R}^n)$ in $L_p^r(w, \mathbb{R}^n)$.

Definition 3.4. Let $1 < p < \infty$, $-n < \varrho < -n/p$ and let $w \in A_{p'}$ with $p' = \frac{p}{p-1}$. Then the spaces $H^\varrho L_p(w, \mathbb{R}^n)$ collect all $h \in S'(\mathbb{R}^n)$ which can be represented as

$$(25) \quad h = \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} h_{J,M} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with } \text{supp } h_{J,M} \subset Q_{J,M},$$

such that

$$(26) \quad \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} < \infty.$$

Furthermore,

$$(27) \quad \|h\|_{H^\varrho L_p(w, \mathbb{R}^n)} \equiv \inf \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)}$$

where the infimum is taken over all representations (25), (26).

Proposition 3.5. Let $1 < p < \infty$, $-n < \varrho < -n/p$, $w \in A_{p'}$ with $p' = \frac{p}{p-1}$ and let $u \equiv -\frac{n}{\varrho}$. Then $1 < u < p$, $w^{1-u} \in A_u$ and

$$(28) \quad H^\varrho L_p(w, \mathbb{R}^n) \hookrightarrow L_{u, w^{1-u}}(\mathbb{R}^n)$$

Furthermore, $D(\mathbb{R}^n)$ is dense in $H^\varrho L_p(w, \mathbb{R}^n)$.

Proof. We observe $1 < u < p$. This implies $p' < u'$ and hence $w \in A_{u'}$ as well as $w^{1-u} \in A_u$. Let $h \in H^\varrho L_p(w, \mathbb{R}^n)$ be optimally represented according to (25)-(27). Hölder's inequality (with respect to the measure $w(x)dx$) yields

$$(29) \quad \begin{aligned} \left\| \tilde{h} \right\|_{L_{u, w^{1-u}}(\mathbb{R}^n)} &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \|h_{J,M}\|_{L_{u, w^{1-u}}(\mathbb{R}^n)} \\ &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \|h_{J,M}\|_{L_{p, w^{1-p}}(\mathbb{R}^n)} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \end{aligned}$$

and, thus, (28) where the first inequality holds by means of the absolute convergence of $\sum_{J,M} h_{J,M}$ in $L_{u, w^{1-u}} \hookrightarrow S'(\mathbb{R}^n)$ and \tilde{h} denotes the representative of $h \in S'(\mathbb{R}^n)$.

It remains to prove that $D(\mathbb{R}^n)$ is dense in $H^\varrho L_p(w, \mathbb{R}^n)$. Let h be given by (25), (26) and let

$$(30) \quad h^L = \sum_{|J| \leq L, |M| \leq L} h_{J,M}, \quad L \in \mathbb{N}.$$

Then

$$\|h - h^L\|_{H^\varrho L_p(w, \mathbb{R}^n)} \rightarrow 0 \quad \text{if } L \rightarrow \infty.$$

Taking into account that $w^{1-p} \in A_p$ any $h_{J,M}$ with $|J| \leq L$, $|M| \leq L$ can be approximated in $L_{p, w^{1-p}}(Q_{J,M})$ by functions belonging to $D(Q_{J,M})$. The sum of these functions approximates h^L and also h . Hence $D(\mathbb{R}^n)$ is dense in $H^\varrho L_p(\mathbb{R}^n)$. \square

Remark 3.6. It follows from (29) that assumption (26) ensures unconditional convergence of (25) in $L_{u,w^{1-u}}(\mathbb{R}^n)$, for $\varrho u = -n$, and hence in $S'(\mathbb{R}^n)$. Moreover, we mention that if one would extend the parameter range of ϱ to $-n < \varrho \leq -n/p$, then the space $H^{-n/p}L_p(w, \mathbb{R}^n)$ would coincide with $L_{p,\tilde{w}}(\mathbb{R}^n)$ where $\tilde{w} \equiv w^{1-p}$ (cf. [RT14, (2.23)-(2.25)] for the unweighted case). Indeed, for $h \in H^{-n/p}L_p(w, \mathbb{R}^n)$ the triangular inequality implies

$$(31) \quad \|h|L_{p,\tilde{w}}(\mathbb{R}^n)\| \leq \|h|H^{-n/p}L_p(w, \mathbb{R}^n)\|.$$

We prove the converse. Let $h \in L_{p,\tilde{w}}(\mathbb{R}^n)$. Let $Q^J = Q_{-J,0_n} = [-2^J, 2^J]^n$, $J \in \mathbb{N}$, be admitted cubes. In dependence on h there is a monotonically increasing sequence of natural numbers $\{J_l\}_{l=0}^\infty$ such that

$$h = \sum_{l=0}^\infty h_l, \quad \text{supp } h_l \subset Q^{J_l}, \quad \|h_l|L_{p,\tilde{w}}(Q^{J_l})\| \leq 2^{-l} \|h|L_{p,\tilde{w}}(\mathbb{R}^n)\|.$$

For suitable J_l one may choose $h_0 = h|Q^{J_0}$ and $h_l = h|Q^{J_l} \setminus Q^{J_{l-1}}$ if $l \in \mathbb{N}$. It is an admitted decomposition in the sense of (25)-(27) (extended to $\varrho = -n/p$). This proves $h \in H^{-n/p}L_p(w, \mathbb{R}^n)$ and the converse of (31) (with an additional factor 2).

4. DUALS AND PREDUALS - THE WEIGHTED CASE

4.1. Predual spaces. The duality with respect to unweighted Morrey spaces is discussed in the scalar case in detail with complete proofs in [RT14] and in the vector-valued case in [RS14]. Here we give complete proofs for Muckenhoupt weighted Morrey spaces following their approach.

Theorem 4.1. *Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$. Then the predual space of $L_p^r(w, \mathbb{R}^n)$ is $H^\varrho L_{p'}(w, \mathbb{R}^n)$. Moreover,*

$$g \in (H^\varrho L_{p'}(w, \mathbb{R}^n))'$$

if, and only if, g can be uniquely represented as

$$(32) \quad g(f) = \int_{\mathbb{R}^n} \tilde{g}(x) f(x) dx$$

for all $f \in D(\mathbb{R}^n)$ where $\tilde{g} \in L_p^r(w, \mathbb{R}^n)$ and

$$(33) \quad \|g|(H^\varrho L_{p'}(w, \mathbb{R}^n))'\| \cong \|\tilde{g}|L_p^r(w, \mathbb{R}^n)\|.$$

Proof. Let $\tilde{g} \in L_p^r(w, \mathbb{R}^n)$. Let $f \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ be optimally represented, that is, we assume that

$$(34) \quad f = \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} h_{J,M} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with } \text{supp } h_{J,M} \subset Q_{J,M},$$

such that

$$\sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}|L_p(\mathbb{R}^n)\| < 2 \|f|H^\varrho L_{p'}(w, \mathbb{R}^n)\|.$$

We observe that by Remark 3.6 the convergence in (34) holds also in $L_{u,w^{1-u}}(\mathbb{R}^n)$ for $\varrho u = -n$. The same holds even for $h \equiv \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} |h_{J,M}|$. That is that $h^N \equiv \sum_{|J|, |M| \leq N} |h_{J,M}|$ tends to h in $L_{u,w^{1-u}}$ for $N \rightarrow \infty$. Therefore exists a partial sum h^{N_l} of h^N which converges pointwise almost everywhere to h for $l \rightarrow \infty$. But we have also that $\sum_{|J|, |M| \leq N_l} h_{J,M}$ converges pointwise almost

everywhere to f (because absolute converging sums are convergent). Lebesgue's monotone convergence theorem, Hölder's inequality and $r + \varrho + n = 0$ yield the estimates

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\tilde{g}(y)f(y)| \, dy = \int_{\mathbb{R}^n} \left| \tilde{g}(y) \lim_{l \rightarrow \infty} \sum_{|J|, |M| \leq N_l} h_{J,M}(y) \right| \, dy \\
& \leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\tilde{g}(y)h_{J,M}(y)| w^{\frac{1}{p}} w^{-\frac{1}{p}} \, dy \\
& \leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|\tilde{g}\|_{L_{p,w}(Q_{J,M})} \\
& \quad \cdot w(Q_{J,M})^{-\left(\frac{1}{p'} + \frac{\varrho}{n}\right)} \|h_{J,M} w^{-\frac{1}{p}}\|_{L_{p'}(\mathbb{R}^n)} \\
& \leq 2 \|\tilde{g}\|_{L_p^r(w, \mathbb{R}^n)} \|f\|_{H^{\varrho} L_{p'}(w, \mathbb{R}^n)}.
\end{aligned}$$

Hence, in particular, any $\tilde{g} \in L_p^r(w, \mathbb{R}^n)$ induces a bounded linear functional on $H^{\varrho} L_{p'}(w, \mathbb{R}^n)$.

Conversely, suppose that g is a bounded linear functional on $H^{\varrho} L_{p'}(w, \mathbb{R}^n)$ with the norm $\|g\|$. Let $\varphi \in D(Q_{J,M})$. Then

$$(35) \quad \|\varphi\|_{H^{\varrho} L_{p'}(w, \mathbb{R}^n)} \leq w(Q_{J,M})^{-\left(\frac{1}{p'} + \frac{\varrho}{n}\right)} \|\varphi w^{-\frac{1}{p}}\|_{L_{p'}(\mathbb{R}^n)} < \infty,$$

where the first inequality holds by Definition 3.4 and the second inequality because of $w^{-\frac{p'}{p}} = w^{1-p'} \in A_{p'}$ and the fact that $D(\mathbb{R}^n)$ is embedded in Muckenhoupt weighted Lebesgue spaces. According to our assumption on g we obtain further

$$|g(\varphi)| \leq \|g\| w(Q_{J,M})^{-\left(\frac{1}{p'} + \frac{\varrho}{n}\right)} \|\varphi w^{-\frac{1}{p}}\|_{L_{p'}(Q_{J,M})}$$

and particularly $g \in (L_{p', w^{1-p'}}(Q_{J,M}))'$. Using the duality $(L_{p', w^{1-p'}}(Q_{J,M}))' \cong L_{p,w}(Q_{J,M})$ we deduce

$$\|g^{Q_{J,M}}\|_{L_{p,w}(Q_{J,M})} \leq \|g\| w(Q_{J,M})^{-\left(\frac{1}{p'} + \frac{\varrho}{n}\right)}.$$

with some $g^{Q_{J,M}} \in L_{p,w}(Q_{J,M})$ such that

$$g(\varphi) = \int_{Q_{J,M}} g^{Q_{J,M}}(x) \varphi(x) \, dx.$$

Taking a sequence of dyadic cubes denoted by Q^l , $l \in \mathbb{N}$, such that $Q^{l-1} \subset Q^l$ and $\bigcup_{l \in \mathbb{N}} Q^l = \mathbb{R}^n$ we get a single function \tilde{g} on \mathbb{R}^n which equals the function $g^{Q_{J,M}}$ on $Q_{J,M}$ and which is in $L_{p,w}(Q_{J,M})$ for all $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$. Moreover, \tilde{g} satisfies then

$$(36) \quad g(\varphi) = \int_{\mathbb{R}^n} \tilde{g}(x) \varphi(x) \, dx$$

for $\varphi \in D(\mathbb{R}^n)$. This proves $\tilde{g} \in L_p^r(w, \mathbb{R}^n)$ and

$$\|\tilde{g}\|_{L_p^r(w, \mathbb{R}^n)} \leq \|g\|$$

by $\frac{n}{p'} + \varrho = -\frac{n}{p} - r$. □

Remark 4.2. An alternative proof of the last result in a more general setting can be found in [ST09].

Proposition 4.3. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$. Then it holds

$$(37) \quad \int_{\mathbb{R}^n} |g(y)f(y)| \, dy \leq \|g|L_p^r(w, \mathbb{R}^n)\| \|f|H^{\varrho}L_{p'}(w, \mathbb{R}^n)\|$$

for $g \in L_p^r(w, \mathbb{R}^n)$ and $f \in H^{\varrho}L_{p'}(w, \mathbb{R}^n)$. Moreover, holds even equality in (33). Furthermore, for $g \in L_p^r(w, \mathbb{R}^n)$ we have

$$(38) \quad \|g|L_p^r(w, \mathbb{R}^n)\| = \sup_f \left| \int_{\mathbb{R}^n} g(x)f(x) \, dx \right|$$

where the supremum is taken over all $f \in D(\mathbb{R}^n)$ with $\|f|H^{\varrho}L_{p'}(w, \mathbb{R}^n)\| \leq 1$.

Proof. Let $\tilde{g} \in L_p^r(w, \mathbb{R}^n)$, $f \in H^{\varrho}L_{p'}(w, \mathbb{R}^n)$ and let be $\varepsilon > 0$. Let $f \in H^{\varrho}L_{p'}(w, \mathbb{R}^n)$ be represented such that

$$f = \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} h_{J,M} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with} \quad \text{supp } h_{J,M} \subset Q_{J,M},$$

such that

$$\sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-\left(\frac{1}{p} + \frac{\varrho}{n}\right)} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} < (1 + \varepsilon) \|f|H^{\varrho}L_p(w, \mathbb{R}^n)\|.$$

Now we deduce the desired statements arguing as in the proof of Theorem 4.1. \square

Remark 4.4. Theorem 4.1 has some history in the unweighted situation (cf. [Lon84, Zor86, Kal98, GR01, AX04, GM13, Ros13, RT14]).

4.2. Dual spaces. In the proof of the next theorem, we benefit from the following general assertion.

Proposition 4.5 (p. 73 of [ET96], Lemma in Section 1.11.1 of [Tri78]). Let $\{A_j\}_{j \in \mathbb{N}_0}$ be a sequence of complex Banach spaces and $\{A'_j\}_{j \in \mathbb{N}_0}$ their respective duals. For

$$\begin{aligned} c_0(\{A_j\}) &\equiv \left\{ a \equiv \{a_j\}_{j \in \mathbb{N}_0} \mid a_j \in A_j, \right. \\ &\quad \left. \|a|c_0(A_j)\| \equiv \|a|\ell_{\infty}(A_j)\| \equiv \sup_j \|a_j|A_j\| < \infty, \|a_j|A_j\| \rightarrow 0 \right\}, \\ \ell_1(\{A'_j\}) &\equiv \left\{ a' \equiv \{a'_j\}_{j \in \mathbb{N}_0} \mid a'_j \in A'_j, \|a'|\ell_1(A'_j)\| \equiv \sum_j \|a_j|A'_j\| < \infty \right\} \end{aligned}$$

it holds

$$\begin{aligned} (c_0(\{A_j\}))' &= \ell_1(\{A'_j\}) \quad \text{with} \quad a'(a) = \sum_{j=0}^{\infty} a'_j(a_j) \quad \text{and} \\ \| \cdot | (c_0(A_j))' \| &= \| \cdot | \ell_1(A'_j) \|. \end{aligned}$$

Lemma 4.6 (§8.6 in [Ste93], p. 39). Let $w \in A_p$ for $1 < p < \infty$. Then $w \notin L_1(\mathbb{R}^n)$.

Proof. Let Q be a cube. Moreover, let us assume $w \in L_1(\mathbb{R}^n)$. Taking into account the reverse doubling condition of w (cf. (11)) we deduce

$$w(Q) \leq c^l w(2^l Q) \leq c^l w(\mathbb{R}^n) < \infty, \quad l \in \mathbb{N}$$

for a constant $c < 1$. Passing to the limit $l \rightarrow \infty$ we obtain $w(Q) = 0$ for arbitrary cubes Q and hence w is a.e. identically zero which contradicts our assumption $w \in A_p$. \square

Proposition 4.7. Let $1 < p < \infty$, $-n < \varrho < -n/p$ and let $w \in A_{p'}$ with $p' = \frac{p}{p-1}$. Then the predual Morrey space $H^{\varrho}L_p(w, \mathbb{R}^n)$ collects all $h \in S'(\mathbb{R}^n)$ which can be represented as

$$(39) \quad h = \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} h_{x,J} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with} \quad \text{supp } h_{x,J} \subset Q(x, J),$$

such that

$$(40) \quad \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{x,J} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} < \infty,$$

where $Q(x, J)$ is defined as in (13). Moreover, we have the equivalence of norms

$$(41) \quad \begin{aligned} \|h\|_{H^{\varrho}L_p(w, \mathbb{R}^n)}^* &\equiv \inf \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{x,J} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\ &\cong \|h\|_{H^{\varrho}L_p(w, \mathbb{R}^n)} \end{aligned}$$

where the infimum is taken over all representations (39), (40).

Proof. Let $h \in S'(\mathbb{R}^n)$ with $\|h\|_{H^{\varrho}L_p(w, \mathbb{R}^n)}^* < \infty$ be optimally represented according to (39)-(41). We define

$$\widetilde{h_{J,M}} \equiv \sum_{x \in \mathbb{Q}^n: x \in Q_{J,M}} h_{x,J}$$

for all $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$. Then $\text{supp } \widetilde{h_{J,M}} \subset 2Q_{J,M}$. The cube $2Q_{J,M}$ is contained in at least one cube $Q_{J-2, \widetilde{M}}$ with an appropriate $\widetilde{M} \in \mathbb{Z}^n$ (cf. (14)). Then we define for $J \in \mathbb{Z}$

$$h_{J-2, \widetilde{M}} \equiv \sum_{M \in \mathbb{Z}^n: 2Q_{J,M} \subset Q_{J-2, \widetilde{M}}} \widetilde{h_{J,M}}, \quad \widetilde{M} \in \mathbb{Z}^n$$

and $\widetilde{h_{J,M}}$ is not summed up in another $h_{J-2, \widetilde{M}}$ with $\widetilde{M} \neq \widetilde{M}$

where we point out that the second condition with respect to the summation in the definition of $h_{J-2, \widetilde{M}}$ ensures that

$$\sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} h_{x,J} = \sum_{J \in \mathbb{Z}, \widetilde{M} \in \mathbb{Z}^n} h_{J-2, \widetilde{M}}.$$

Finally,

$$\begin{aligned} &\sum_{J \in \mathbb{Z}, \widetilde{M} \in \mathbb{Z}^n} w(Q_{J-2, \widetilde{M}})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J-2, \widetilde{M}} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\ &\leq \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{x,J} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \leq 2 \|h\|_{H^{\varrho}L_p(w, \mathbb{R}^n)}^* \end{aligned}$$

where the last but one inequality holds by the fact $Q(x, J) \subset 2Q_{J,M}$ for $x \in Q_{J,M}$. \square

Theorem 4.8. *Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$. Then the dual space of $\mathring{L}_p^r(w, \mathbb{R}^n)$ is $H^\varrho L_{p'}(w, \mathbb{R}^n)$. Moreover, $g \in \left(\mathring{L}_p^r(w, \mathbb{R}^n)\right)'$ if, and only if, g can be uniquely represented as*

$$(42) \quad g(f) = \int_{\mathbb{R}^n} \tilde{g}(x) f(x) dx$$

for all $f \in L_{-\frac{n}{r}, w}(\mathbb{R}^n)$ where $\tilde{g} \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ and

$$(43) \quad \left\| g \left| \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)' \right| \right\| \cong \|\tilde{g}\|_{H^\varrho L_{p'}(w, \mathbb{R}^n)}.$$

Proof. It follows from (37) that any $g \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ induces a bounded linear functional on $\mathring{L}_p^r(w, \mathbb{R}^n)$.

Conversely, suppose that g is a bounded linear functional on $\mathring{L}_p^r(w, \mathbb{R}^n)$ with norm $\|g\|$. Let $f \in D(\mathbb{R}^n)$ and $f_{x,J} = f \chi_{Q(x,J)}$ for $J \in \mathbb{Z}$ and $x \in \mathbb{Q}^n$. Then

$$(44) \quad \begin{aligned} & w(Q(x, J))^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|f_{x,J}\|_{L_{p,w}(Q(x, J))} \\ & \leq w(Q(x, J))^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|f\|_{L_{p,w}(\mathbb{R}^n)} \rightarrow 0 \end{aligned}$$

if $J \rightarrow -\infty$. This follows from $\frac{n}{p} + r > 0$, Lemma 4.6 and Lebesgue's monotone convergence theorem. Moreover,

$$(45) \quad w(Q(x, J))^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|f_{x,J}\|_{L_{p,w}(Q(x, J))} \leq w(Q(x, J))^{-\frac{r}{n}} \|f\|_{L_\infty(\mathbb{R}^n)} \rightarrow 0$$

if $J \rightarrow \infty$ follows from $r < 0$ and $w(Q(x, J)) \rightarrow 0$ for $J \rightarrow \infty$ (which holds by Lebesgue's dominated convergence theorem and $w \in L_1^{\text{loc}}(\mathbb{R}^n)$). Furthermore, for fixed $J_0 \in \mathbb{N}$ and $|J| \leq J_0$ we have

$$(46) \quad w(Q(x, J))^{-\left(\frac{1}{p} + \frac{r}{n}\right)} \|f_{x,J}\|_{L_{p,w}(Q(x, J))} \rightarrow 0 \quad \text{if } |x| \rightarrow \infty,$$

since $\text{supp}(f) \cap Q(x, -J_0) = \emptyset$ for $|x| > l$ and $l = l(f, x, J_0) \in \mathbb{N}$ being sufficiently large. Using Lemma 2.5 we observe further that

$$(47) \quad \begin{aligned} \|f\|_{L_p^r(w, \mathbb{R}^n)} & \cong \sup_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} \left(\int_{Q(x, J)} |f(y)|^p w(Q(x, J))^{-\left(1 + \frac{rp}{n}\right)} w(y) dy \right)^{\frac{1}{p}} \\ & = \|F\|_{c_0(L_{p, \mu_{x,J}}(Q(x, J)))} \end{aligned}$$

where $F \equiv \{f_{x,J}\}_{x \in \mathbb{Q}^n, J \in \mathbb{Z}}$, $\mu_{x,J}(dy) \equiv w(Q(x, J))^{-\left(1 + \frac{rp}{n}\right)} w(y) dy$ and

$$\begin{aligned} & \|F\|_{c_0(L_{p, \mu_{x,J}}(Q(x, J)))} \\ & \equiv \sup_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} \left(\int_{Q(x, J)} |f_{x,J}(y)|^p w(Q(x, J))^{-\left(1 + \frac{rp}{n}\right)} w(y) dy \right)^{\frac{1}{p}}. \end{aligned}$$

Combining (44)-(47) it follows that $\mathring{L}_p^r(w, \mathbb{R}^n)$ is isomorphic to a closed subspace of $c_0(L_{p, \mu_{x,J}}(Q(x, J)))$. More precisely, we have a linear, surjective map $I : f \mapsto \{f_{x,J}\}_{x,J}$ from $\mathring{L}_p^r(w, \mathbb{R}^n)$ onto the closed subspace $\{\{f_{x,J}\} | f \in \mathring{L}_p^r(w, \mathbb{R}^n)\}$ of $c_0(L_{p, \mu_{x,J}}(Q(x, J)))$ satisfying (47),

$$I \mathring{L}_p^r(w, \mathbb{R}^n) = \{\{f_{x,J}\}_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} | f \in \mathring{L}_p^r(w, \mathbb{R}^n)\} \hookrightarrow c_0(L_{p, \mu_{x,J}}(Q(x, J))).$$

Hahn-Banach's theorem yields $g \in \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)'$ if, and only if,
 $g \in (c_0 (L_{p, \mu_{x, J}}(Q(x, J))))'$ and by Proposition 4.5 we have the representation

$$(48) \quad g(f) = \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} \int_{Q(x, J)} f(y) g_{x, J}(y) w(Q(x, J))^{-(1+\frac{rp}{n})} dy$$

for any $f \in \mathring{L}_p^r(w, \mathbb{R}^n)$ with $\{g_{x, J}\} \in \ell_1 (L_{p', \widetilde{\mu_{x, J}}} (Q(x, J)))$ where

$$\begin{aligned} & \|\{g_{x, J}\} | \ell_1 (L_{p', \widetilde{\mu_{x, J}}} (Q(x, J)))\| \\ &= \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} \left(\int_{Q(x, J)} |g_{x, J}(y)|^{p'} w(Q(x, J))^{-(1+\frac{rp}{n})} w(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}} \end{aligned}$$

and $\widetilde{\mu_{x, J}}(dy) \equiv w(Q(x, J))^{-(1+\frac{rp}{n})} w(y)^{-\frac{p'}{p}} dy$. To get (48) we used

$$(L_{p, \mu_{x, J}}(Q(x, J)))' = L_{p', \widetilde{\mu_{x, J}}} (Q(x, J)) \quad , x \in \mathbb{Q}^n, J \in \mathbb{Z},$$

in order to apply Proposition 4.5 which is deduced analogously to the well-known duality assertion in Muckenhoupt weighted Lebesgue spaces $(L_{p, w}(Q(x, J)))' = L_{p', w^{1-p'}}(Q(x, J))$ (where $w \in A_p$) replacing the Lebesgue measure by the Lebesgue measure multiplied with the constant $w(Q(x, J))^{-(1+\frac{rp}{n})}$. Moreover, Hahn-Banach's theorem and Lemma 2.5 enable us to assume

$$(49) \quad \left\| g \left| \left(\mathring{L}_p^r(\mathbb{R}^n) \right)' \right. \right\| \cong \|\{g_{x, J}\} | \ell_1 (L_{p', \widetilde{\mu_{x, J}}} (Q(x, J)))\|.$$

Using Lebesgue's dominated convergence theorem (cf. (51) for an integrable majorant) we deduce from (48) the representation

$$(50) \quad g(f) = \int_{\mathbb{R}^n} f(y) \widetilde{g}(y) dy$$

for any $f \in D(\mathbb{R}^n)$ and

$$\widetilde{g}(y) \equiv \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} g_{x, J}(y) \chi_{Q(x, J)}(y) w(Q(x, J))^{-(1+\frac{rp}{n})}.$$

With

$$h_{x, J} = g_{x, J}(y) \chi_{Q(x, J)}(y) w(Q(x, J))^{-(1+\frac{rp}{n})}$$

and $r + \varrho = -n$ one has

$$\begin{aligned} & w(Q(x, J))^{-(\frac{1}{p'} + \frac{\varrho}{n})} \left(\int_{Q(x, J)} |h_{x, J}(y)|^{p'} w(y)^{-\frac{p'}{p}} dy \right)^{1/p'} \\ &= w(Q(x, J))^{\frac{1}{p} + \frac{r}{n}} \left(\int_{Q(x, J)} |g_{x, J}(y)|^{p'} w(Q(x, J))^{-(1+\frac{rp}{n})(p'-1)} \right. \\ & \quad \cdot \left. w(Q(x, J))^{-(1+\frac{rp}{n})} w(y)^{-\frac{p'}{p}} dy \right)^{1/p'} \\ &= \left(\int_{Q(x, J)} |g_{x, J}(y)|^{p'} w(Q(x, J))^{-(1+\frac{rp}{n})} w(y)^{-\frac{p'}{p}} dy \right)^{1/p'}. \end{aligned}$$

Using Proposition 4.7 and (49) we see therefore that

$$\|\widetilde{g} | H^{\varrho} L_{p'}(w, \mathbb{R}^n)\| \leq c_1 \|\{g_{x, J}\} | \ell_1 (L_{p', \widetilde{\mu_{x, J}}} (Q(x, J)))\| \leq c_2 \|g\|$$

where $\tilde{g} = \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} h_{x,J}$ converges in $L_{u,w^{1-u}}(\mathbb{R}^n)$ with $\varrho u = -n$ by Proposition 3.5 and hence in $S'(\mathbb{R}^n)$ (noting that $1 < u < p'$ and $w^{1-u} \in A_u$). We can establish an integrable majorant to justify (50) (resp. the application of Lebesgue's dominated convergence theorem) by the same argumentation using $f \in D(\mathbb{R}^n)$. Indeed,

$$(51) \quad |f| \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} |h_{x,J}| = \sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} |f h_{x,J}|$$

is integrable by Hölder's inequality since $\sum_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} |h_{x,J}| \in H^\varrho L_{p'}(w, \mathbb{R}^n) \hookrightarrow L_{u,w^{1-u}}(\mathbb{R}^n)$, $\varrho u = -n$, analogously to $\tilde{g} \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ and (28) and moreover $f \in L_{u',w}(\mathbb{R}^n)$ where $w \in A_{u'}$ by $p < u'$. It remains to show that \tilde{g} represents g for all $f \in L_{-\frac{n}{r},w}(\mathbb{R}^n)$ cf. (42) and not only for all $f \in D(\mathbb{R}^n)$. Let $f \in L_{-\frac{n}{r},w}(\mathbb{R}^n)$. Then there is a sequence $f_n \in D(\mathbb{R}^n)$, $n \in \mathbb{N}$, with $f_n \rightarrow f$ almost everywhere and $f_n \rightarrow f$ in $L_{-\frac{n}{r},w}(\mathbb{R}^n)$ and moreover such that f can be dominated by f_n , i.e. $|f_n| \leq |f|$. Because of the continuity of g , (42) for $f_n \in D(\mathbb{R}^n)$ and Lebesgue's dominated convergence theorem ($|\tilde{g}(\cdot)|f(\cdot)$ is an integrable majorant by (37)) we obtain

$$\begin{aligned} g(f) &= \lim_{n \in \mathbb{N}} g(f_n) = \lim_{n \in \mathbb{N}} \int_{\mathbb{R}^n} \tilde{g}(x) f_n(x) dx = \int_{\mathbb{R}^n} \tilde{g}(x) \lim_{n \in \mathbb{N}} f_n(x) dx \\ &= \int_{\mathbb{R}^n} \tilde{g}(x) f(x) dx \end{aligned}$$

and hence (42) for all $f \in L_{-\frac{n}{r},w}(\mathbb{R}^n)$. \square

Proposition 4.9. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$. Using the norms

$$(52) \quad \|f|L_p^r(w, \mathbb{R}^n)\|^* \equiv \sup_{x \in \mathbb{Q}^n, J \in \mathbb{Z}} w(Q(x, J))^{-(\frac{1}{p} + \frac{n}{r})} \|f|L_{p,w}(Q(x, J))\|$$

(cf. (13)) and $\|\cdot\|^{H^\varrho L_{p'}(w, \mathbb{R}^n)} \|^*$ (cf. (40)) we obtain even equality in (43). That is, $g \in \left(\mathring{L}_p^r(w, \mathbb{R}^n)\right)'$ if, and only if, g can be uniquely represented as

$$g(f) = \int_{\mathbb{R}^n} \tilde{g}(x) f(x) dx$$

for all $f \in L_{-\frac{n}{r},w}(\mathbb{R}^n)$ where $\tilde{g} \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ and

$$\left\| g \left| \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)' \right. \right\|^* \equiv \sup_{\substack{f \in \mathring{L}_p^r(w, \mathbb{R}^n): \\ \|f|L_p^r(w, \mathbb{R}^n)\|^* \leq 1}} |g(f)| = \|\tilde{g}|H^\varrho L_{p'}(w, \mathbb{R}^n)\|^*.$$

Moreover, for $\tilde{g} \in H^\varrho L_{p'}(w, \mathbb{R}^n)$ it holds

$$(53) \quad \|\tilde{g}|H^\varrho L_{p'}(w, \mathbb{R}^n)\|^* = \sup_f \left| \int_{\mathbb{R}^n} \tilde{g}(x) f(x) dx \right|$$

where the supremum is taken over all $f \in L_{-\frac{n}{r},w}(\mathbb{R}^n)$ with $\|f|L_p^r(w, \mathbb{R}^n)\|^* \leq 1$.

Proof. Analogously to the arguments establishing (37) we get

$$\left| \int_{\mathbb{R}^n} f(y) \tilde{g}(y) dy \right| \leq \|f|L_p^r(w, \mathbb{R}^n)\|^* \|\tilde{g}|H^\varrho L_{p'}(w, \mathbb{R}^n)\|^*$$

for $f \in \mathring{L}_p^r(w, \mathbb{R}^n)$ and $\tilde{g} \in H^\varrho L_{p'}(w, \mathbb{R}^n)$. This yields for the induced functional g of \tilde{g}

$$\left\| g \left| \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)' \right| \right\|^* \leq \|\tilde{g}\|_{H^\varrho L_{p'}(w, \mathbb{R}^n)}^*.$$

Conversely, suppose g is a bounded linear functional on $\mathring{L}_p^r(w, \mathbb{R}^n)$ with norm $\left\| g \left| \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)' \right| \right\|$. Then we show by the same arguments as in the proof of Theorem 4.8 that g can be represented as in (42) where \tilde{g} also satisfies

$$\|\tilde{g}\|_{H^\varrho L_{p'}(w, \mathbb{R}^n)}^* \leq \left\| g \left| \left(\mathring{L}_p^r(w, \mathbb{R}^n) \right)' \right| \right\|^*.$$

□

Remark 4.10. The unweighted case of Theorem 4.8 is stated in [AX12] and proved in [Ros13, RT14]. In dimension $n = 1$ it is also proved in a periodic setting on the torus in [SY14]. Its vector-valued version is investigated in [RS14].

Corollary 4.11. (i) Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$ and let $w \in A_p$. Then

$\mathring{L}_p^r(w, \mathbb{R}^n)$ and $L_p^r(w, \mathbb{R}^n)$ are Banach spaces.

(ii) Let $1 < p < \infty$, $-n < \varrho < -n/p$ and let $w \in A_{p'}$ with $p' = \frac{p}{p-1}$. Then $H^\varrho L_p(w, \mathbb{R}^n)$ are Banach spaces.

Proof. The spaces $L_p^r(w, \mathbb{R}^n)$ and $H^\varrho L_p(w, \mathbb{R}^n)$ are duals of normed vector spaces $(H^\varrho L_p(w, \mathbb{R}^n)$ resp. $\mathring{L}_p^r(w, \mathbb{R}^n)$) and therefore complete. Since $\mathring{L}_p^r(w, \mathbb{R}^n)$ is a closed subspace of the complete space $L_p^r(w, \mathbb{R}^n)$ we get also the completeness of $\mathring{L}_p^r(w, \mathbb{R}^n)$. □

Corollary 4.12. Let $1 < p < \infty$, $-n < \varrho < -\frac{n}{p}$ and let $w \in A_{p'}$ with $p' = \frac{p}{p-1}$.

(i) If $|f| \leq |g|$ almost everywhere, then $\|f\|_{H^\varrho L_p(w, \mathbb{R}^n)} \leq c \|g\|_{H^\varrho L_p(w, \mathbb{R}^n)}$ where the constant c is independent of f and g .

(ii) Moreover, it holds $\|f\|_{H^\varrho L_p(w, \mathbb{R}^n)} \cong \| |f| \|_{H^\varrho L_p(w, \mathbb{R}^n)}$.

Proof. Let $|f| \leq |g|$ almost everywhere. By (53) we have

$$\begin{aligned} \|f\|_{H^\varrho L_p(w, \mathbb{R}^n)}^* &\leq \sup_h \int_{\mathbb{R}^n} |f(x)| |h(x)| dx \leq \sup_h \int_{\mathbb{R}^n} |f(x)| h(x) dx \\ &\leq \sup_h \int_{\mathbb{R}^n} |g(x)| h(x) dx = \| |g| \|_{H^\varrho L_p(w, \mathbb{R}^n)}^* \end{aligned}$$

where the supremum is taken over all $h \in L_{-\frac{p}{\varrho}, w}(\mathbb{R}^n)$ with $\|h\|_{L_p^r(w, \mathbb{R}^n)}^* \leq 1$ and which implies

$$(54) \quad \|f\|_{H^\varrho L_p(w, \mathbb{R}^n)} \leq c_1 \| |f| \|_{H^\varrho L_p(w, \mathbb{R}^n)} \leq c_2 \| |g| \|_{H^\varrho L_p(w, \mathbb{R}^n)}$$

Let $f \in H^\varrho L_p(w, \mathbb{R}^n)$ be optimally represented, that is, we assume that

$$f = \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} h_{J,M} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with} \quad \text{supp } h_{J,M} \subset Q_{J,M},$$

such that

$$\sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\rho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} < 2 \|f\|_{H^\rho L_p(w, \mathbb{R}^n)}.$$

Thus,

$$|f| \leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} |h_{J,M}| \equiv \bar{g}$$

and

$$\begin{aligned} \|\bar{g}\|_{H^\rho L_p(w, \mathbb{R}^n)} &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\rho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\ &< 2 \|f\|_{H^\rho L_p(w, \mathbb{R}^n)}. \end{aligned}$$

Using (54) it follows

$$\| |f| \|_{H^\rho L_p(w, \mathbb{R}^n)} \leq c \|f\|_{H^\rho L_p(w, \mathbb{R}^n)}$$

and therefore (ii). Taking again into account (54) we obtain finally (i). \square

5. THE EXTRAPOLATION TO MORREY SPACES

Definition 5.1. The *Hardy-Littlewood maximal operator* M is given by

$$(Mf)(y) \equiv \sup_{Q: y \in Q} \frac{1}{|Q|} \int_Q |f(z)| dz, \quad f \in L_1^{\text{loc}}(\mathbb{R}^n)$$

where the supremum is taken over all cubes Q (whose sides are parallel to the coordinate axes) in \mathbb{R}^n which contain $y \in \mathbb{R}^n$.

The following proposition establishes the boundedness of the maximal operator in the predual Morrey spaces. We will use this fact for the extrapolation result appearing in Theorem 5.3 below. Only the boundedness of the Hardy-Littlewood maximal operator (in the predual Morrey spaces) is required for its proof beside the duality. By the fact that the same arguments even give boundedness results of some singular integrals (in the predual Morrey spaces) we formulate the result in a more general version.

Proposition 5.2. Let $1 < p < \infty$, $-n < \rho < -\frac{n}{p}$ and let $w \in A_{p'}$ with $p' = \frac{p}{p-1}$. Let T be an operator such that it holds

$$(55) \quad T : L_{p, w^{1-p}}(\mathbb{R}^n) \hookrightarrow L_{p, w^{1-p}}(\mathbb{R}^n).$$

Moreover, T satisfies the representation formula

$$(56) \quad |(Tf)(y)| \leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{|y-x|^n} dx$$

for all $f \in L_{p, w^{1-p}}(\mathbb{R}^n)$ compactly supported where $y \notin \text{supp}(f)$. Furthermore, we assume that the operator T is

- (1) either linear
- (2) or satisfy

$$(57) \quad \begin{aligned} (T(f_1 + f_2))(y) &\leq (Tf_1)(y) + (Tf_2)(y), \\ (Tf)(y) &= (T(-f))(y) \end{aligned}$$

for $f, f_1, f_2 \in L_{p, w^{1-p}}(\mathbb{R}^n)$ and almost all $y \in \mathbb{R}^n$.

Then there is an unique continuous and bounded extension of \tilde{T} of T to $H^\varrho L_p(w, \mathbb{R}^n)$ such that

$$\tilde{T} : H^\varrho L_p(w, \mathbb{R}^n) \hookrightarrow H^\varrho L_p(w, \mathbb{R}^n)$$

for every $-n < \varrho < -n/p$.

Proof. Let $f \in D(\mathbb{R}^n) \hookrightarrow H^\varrho L_p(w, \mathbb{R}^n)$ be optimally represented, that is, we assume that

$$(58) \quad f = \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} h_{J,M} \quad \text{in } S'(\mathbb{R}^n) \quad \text{with} \quad \text{supp } h_{J,M} \subset Q_{J,M},$$

such that

$$\sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} < 2 \|f\|_{H^\varrho L_p(w, \mathbb{R}^n)}.$$

Without loss of generality we can assume that the convergence in (58) is even pointwise almost everywhere as explained at the beginning of the proof of Theorem 4.1 (and as it holds for a partial sum of it). Thus, using (57) we observe that

$$(59) \quad \begin{aligned} |T(f)| &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} |T(h_{J,M})| \left(\chi_{2Q_{J,M}} + \sum_{l \in \mathbb{N}} \chi_{2^{l+1}Q_{J,M} \setminus 2^l Q_{J,M}} \right) \\ &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \chi_{2Q_{J,M}} |T(h_{J,M})| + \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \sum_{l \in \mathbb{N}} \chi_{2^{l+1}Q_{J,M} \setminus 2^l Q_{J,M}} |T(h_{J,M})| \end{aligned}$$

We obtain by (55) and (10)

$$(60) \quad \begin{aligned} &\left\| \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \chi_{2Q_{J,M}} |T(h_{J,M})| \right\|_{H^\varrho L_p(w, \mathbb{R}^n)} \\ &\leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(2Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|\chi_{2Q_{J,M}} |T(h_{J,M})| w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\ &\leq c_1 \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|h_{J,M} w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\ &\leq c_2 \|f\|_{H^\varrho L_p(w, \mathbb{R}^n)}. \end{aligned}$$

Furthermore, (56) and Hölder's inequality yield

$$\begin{aligned} \chi_{2^{l+1}Q_{J,M} \setminus 2^l Q_{J,M}} |T(h_{J,M})| &\leq \frac{c}{|2^l Q_{J,M}|} \int_{Q_{J,M}} |h_{J,M}(x)| dx \\ &\leq \frac{c}{|2^l Q_{J,M}|} \left\| |h_{J,M}| w^{-\frac{1}{p'}} \right\|_{L_p(\mathbb{R}^n)} w(Q_{J,M})^{\frac{1}{p'}} \end{aligned}$$

Using $w \in A_{p'}$, (11) and $1 + \frac{\varrho}{n} > 0$ this implies

$$\begin{aligned}
(61) \quad & \left\| \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \sum_{l \in \mathbb{N}} \chi_{2^{l+1}Q_{J,M} \setminus 2^l Q_{J,M}} |T(h_{J,M})| \right\|_{H^{\varrho} L_p(w, \mathbb{R}^n)} \\
& \leq \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \sum_{l \in \mathbb{N}} w(2^{l+1}Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \|\chi_{2^{l+1}Q_{J,M} \setminus 2^l Q_{J,M}} |T(h_{J,M})| w^{-\frac{1}{p'}}\|_{L_p(\mathbb{R}^n)} \\
& \leq c_1 \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \sum_{l \in \mathbb{N}} w(2^{l+1}Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \\
& \quad \cdot \frac{w(Q_{J,M})^{\frac{1}{p'}}}{|2^l Q_{J,M}|} \left\| |h_{J,M}| w^{-\frac{1}{p'}} \right\|_{L_p(\mathbb{R}^n)} w^{1-p} (2^{l+1}Q_{J,M})^{\frac{1}{p}} \\
& \leq c_2 \sum_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{N}} \frac{w(Q_{J,M})^{\frac{1}{p'} + \frac{1}{p} + \frac{\varrho}{n}}}{w(2^{l+1}Q_{J,M})^{\frac{1}{p'} + \frac{1}{p} + \frac{\varrho}{n}}} \right) w(Q_{J,M})^{-(\frac{1}{p} + \frac{\varrho}{n})} \left\| |h_{J,M}| w^{-\frac{1}{p'}} \right\|_{L_p(\mathbb{R}^n)} \\
& \leq c_3 \|f\|_{H^{\varrho} L_p(w, \mathbb{R}^n)}.
\end{aligned}$$

Finally, the assertion holds by (59), Corollary 4.12, (60), (61) and continuous and bounded extension taking into account $D(\mathbb{R}^n) \hookrightarrow H^{\varrho} L_p(w, \mathbb{R}^n)$ dense. Here we mention that whenever T is not linear the continuous and bounded extension is derived as in (70) below. \square

Theorem 5.3. *Assume that for some family \mathcal{F} of ordered pairs of non-negative locally integrable functions (g, f) , for some $1 < p_1 < \infty$ and every $w \in A_{p_1}$ we have*

$$(62) \quad \|g\|_{L_{p_1, w}(\mathbb{R}^n)} \leq c_1 \|f\|_{L_{p_1, w}(\mathbb{R}^n)} \quad \text{for all } (g, f) \in \mathcal{F}.$$

Then for every $1 < p < \infty$, every $-\frac{n}{p} \leq r < 0$ and every $w \in A_p$ we have

$$(63) \quad \|g\|_{L_p^r(w, \mathbb{R}^n)} \leq c_2 \|f\|_{L_p^r(w, \mathbb{R}^n)}$$

for all $(g, f) \in \mathcal{F}$. The constants c_1 and c_2 in (62) and (63) do not depend on (f, g) but may depend on w , p_1 and p .

Proof. If $w \in A_p$, then there exists a $p_0 \in (1, p)$ such that even $w \in A_{p/p_0}$. Therefore the space $L_{\frac{p}{p_0}}^{rp_0}(w, \mathbb{R}^n)$ is well-defined and the Hardy-Littlewood maximal operator is bounded on its predual $H^{\varrho} L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n)$ where $\varrho \equiv -n - rp_0$ by Proposition 5.2. Fix $(g, f) \in \mathcal{F}$. By the duality representation (38) we have

$$\begin{aligned}
\|g\|_{L_p^r(w, \mathbb{R}^n)}^{p_0} &= \|g^{p_0}\|_{L_{\frac{p}{p_0}}^{rp_0}(w, \mathbb{R}^n)} = \sup_h \left| \int_{\mathbb{R}^n} g^{p_0}(x) h(x) dx \right| \\
&\leq \sup_h \int_{\mathbb{R}^n} g^{p_0}(x) |h(x)| dx
\end{aligned}$$

where the supremum is taken over all $h \in D(\mathbb{R}^n)$ with $\left\| |h| H^{\varrho} L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \leq 1$. Therefore, it will suffice to fix such a function h and show that

$$\int_{\mathbb{R}^n} g^{p_0}(x) |h(x)| dx \leq c \|f\|_{L_p^r(w, \mathbb{R}^n)}^{p_0}$$

for a constant c which does not depend on h . Hence, we apply the Rubio de Francia algorithm defining

$$(R|h|)(x) = \sum_{k=0}^{\infty} \frac{(M^k|h|)(x)}{\left(2 \left\| M|H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \rightarrow H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \right)^k}$$

using the predefinition that M^0 stands for the identity. Then we have

- (i) $|h(x)| \leq (R|h|)(x)$ almost everywhere,
- (ii) $\left\| (R|h|)(\cdot) |H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \leq 2 \left\| |h| |H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\|$ and
- (iii) $(M(R|h|))(x) \leq 2 \left\| M|H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \rightarrow H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| (R|h|)(x)$ almost everywhere.

For it we assume $f \in L_p^r(w, \mathbb{R}^n)$ (otherwise (63) is trivial). By (i) we have that

$$\int_{\mathbb{R}^n} g^{p_0}(x) |h(x)| dx \leq \int_{\mathbb{R}^n} g^{p_0}(x) R(|h|)(x) dx.$$

By reason of (iii), $A_1 \subset A_{p_0}$ and the extrapolation in the usual Lebesgue weighted spaces using (62) (cf. [Duo11, CMP11, Duo13]) we obtain

$$\int_{\mathbb{R}^n} g^{p_0}(x) R(|h|)(x) dx \leq c \int_{\mathbb{R}^n} f^{p_0}(x) R(|h|)(x) dx$$

where the constant c does not depend on (g, f) (but may depend on p_1, p_0 and w). By the generalized Hölder's inequality (37) and (ii),

$$\begin{aligned} \int_{\mathbb{R}^n} f^{p_0}(x) R(|h|)(x) dx &\leq \left\| f^{p_0} |L_{\frac{p}{p_0}}^{rp_0}(w, \mathbb{R}^n) \right\| \left\| R(|h|) |H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \\ &\leq 2 \left\| f |L_p^r(w, \mathbb{R}^n) \right\|^{p_0} \left\| |h| |H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \\ &\leq 2c \left\| f |L_p^r(w, \mathbb{R}^n) \right\|^{p_0} \left\| |h| |H^e L_{(\frac{p}{p_0})}'(w, \mathbb{R}^n) \right\| \end{aligned}$$

where the last inequality follows by Corollary 4.12. \square

Remark 5.4. The proof adapts the results of [CMP11, Thm. 4.6] and [CGCMP06] for Morrey spaces (which are not Banach function spaces) and shows that their approach for Banach function spaces using associated spaces could be used also with the dual or predual and a density relation instead of associated spaces for function spaces which are not Banach function spaces. The extrapolation result (1) \Rightarrow (2) is stated here in pairs of functions free of operators. This setting has several advantages. For example, it allows to derive vector-valued inequalities without any further effort (cf. [CMP04, CMP11, Duo13]). We proved (3) in this general version.

The unweighted situation of Proposition 5.2 is proved in [IST15]. We want to mention that if one wants to derive the boundedness of some singular integral using Proposition 5.2, e.g. the Hilbert transform, one needs for satisfying (56) that already its maximally truncations are studied in the appropriate weighted Lebesgue space (cf. (55)).

However, beside its preduals the maximal operator is also bounded in the weighted Morrey spaces as introduced in Definition 2.2 (cf. [KS09, Thm. 3.2], [Mus12, Rem. 3], [Ros13, Thm. 1.6.3] and [KGS14, Cor. 4.2]). For modifications of weighted

Morrey spaces we refer to [Sam14, Sam13, RSS13, ST05] and the references given there.

6. MAPPING PROPERTIES OF OPERATORS

6.1. The main theorem.

Theorem 6.1. *Let T be an operator and suppose that for some $p_0 \in (1, \infty)$ and for every $w \in A_{p_0}$ it holds*

$$(64) \quad T : L_{p_0, w}(\mathbb{R}^n) \hookrightarrow L_{p_0, w}(\mathbb{R}^n).$$

Moreover, assume that the operator T is

- (1) either linear
- (2) or satisfy

$$(65) \quad \begin{aligned} (T(f_1 + f_2))(y) &\leq (Tf_1)(y) + (Tf_2)(y), \\ (Tf)(y) &= (T(-f))(y) \end{aligned}$$

for $f, f_1, f_2 \in D(\mathbb{R}^n)$ and almost all $y \in \mathbb{R}^n$.

Then, the following statements hold true.

- (1) There are unique continuous and bounded extensions \tilde{T} of T to $\mathring{L}_p^r(w, \mathbb{R}^n)$ such that

$$\tilde{T} : \mathring{L}_p^r(w, \mathbb{R}^n) \hookrightarrow \mathring{L}_p^r(w, \mathbb{R}^n)$$

for every $1 < p < \infty$, every $-\frac{n}{p} < r < 0$ and every $w \in A_p$.

- (2) If T is furthermore linear, then there are linear and bounded extensions \tilde{T} of T to $L_p^r(w, \mathbb{R}^n)$ such that

$$\tilde{T} : L_p^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n)$$

for every $1 < p < \infty$, every $-\frac{n}{p} < r < 0$ and every $w \in A_p$.

- (3) Let $1 < p' < \infty$, $-n < \varrho < -\frac{n}{p'}$ and let $w \in A_p$ with $p = \frac{p'}{p'-1}$. If T is furthermore linear, then the dual operator of the unique linear and bounded extension \tilde{T} of T acting in $\mathring{L}_p^r(w, \mathbb{R}^n)$ satisfies

$$\tilde{T}' : H^\varrho L_{p'}(w, \mathbb{R}^n) \hookrightarrow H^\varrho L_{p'}(w, \mathbb{R}^n).$$

If T is in addition formally self-adjoint with respect to (64), i.e. if we assume that

$$(66) \quad \langle Tf, g \rangle_{(L_{p_0, w}, L_{p'_0, w'})} = \langle f, Tg \rangle_{(L_{p_0, w}, L_{p'_0, w'})} \quad \text{for all } f, g \in D(\mathbb{R}^n)$$

where $w' \equiv w^{1-p'_0}$, then \tilde{T}' is the unique linear and bounded extension of T acting in $H^\varrho L_{p'}(w, \mathbb{R}^n)$.

Proof. Using (64) Theorem 5.3 with $\mathcal{F} \equiv \{(|Tf|, |f|) \mid f \in D(\mathbb{R}^n)\}$ yields

$$(67) \quad \|Tf\|_{L_p^r(w, \mathbb{R}^n)} \leq c \|f\|_{L_p^r(w, \mathbb{R}^n)}$$

for all $f \in D(\mathbb{R}^n)$, where the constant c does not depend on f . Hence, $T : D(\mathbb{R}^n) \rightarrow L_p^r(w, \mathbb{R}^n)$. Theorem 5.3 as well as well-known extrapolation techniques cf. [CMP11, Thm. 1.4] imply

$$(68) \quad \|Tf\|_{L_{p, w}(\mathbb{R}^n)} \leq c \|f\|_{L_{p, w}(\mathbb{R}^n)}$$

for all $1 < p < \infty$ where the constant c does not depend on $f \in D(\mathbb{R}^n)$ and therefore $T : D(\mathbb{R}^n) \rightarrow L_{-\frac{n}{p},w}(\mathbb{R}^n)$. Because of the embedding $L_{-\frac{n}{p},w}(\mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n)$ and the density of $D(\mathbb{R}^n)$ in $L_{-\frac{n}{p},w}(\mathbb{R}^n)$ we even have $T : D(\mathbb{R}^n) \rightarrow \mathring{L}_p^r(w, \mathbb{R}^n)$. Indeed, let $f \in D(\mathbb{R}^n)$. Then $Tf \in L_{-\frac{n}{p},w}(\mathbb{R}^n)$ by (68) and thus there is a sequence of functions of $D(\mathbb{R}^n)$ which tends to Tf in $L_{-\frac{n}{p},w}(\mathbb{R}^n)$ and hence in $L_p^r(w, \mathbb{R}^n)$ which shows $Tf \in \mathring{L}_p^r(w, \mathbb{R}^n)$. Thus, if T is linear, then it holds for the unique (linear) continuous and bounded extension \tilde{T} of T

$$(69) \quad \tilde{T} : \mathring{L}_p^r(w, \mathbb{R}^n) \hookrightarrow \mathring{L}_p^r(w, \mathbb{R}^n).$$

If T (is not linear but) satisfies (65), then we have

$$|(Tf_1)(y) - (Tf_2)(y)| \leq (T(f_1 - f_2))(y)$$

for all $f_1, f_2 \in D(\mathbb{R}^n)$ and almost all $y \in \mathbb{R}^n$ and hence in combination with (67)

$$(70) \quad \begin{aligned} \|Tf_1 - Tf_2\|_{L_p^r(w, \mathbb{R}^n)} &\leq \|T(f_1 - f_2)\|_{L_p^r(w, \mathbb{R}^n)} \\ &\leq c \|f_1 - f_2\|_{L_p^r(w, \mathbb{R}^n)}. \end{aligned}$$

Therefore, T is (Lipschitz-)continuous and we get the unique continuous and bounded extension $\tilde{T} : \mathring{L}_p^r(w, \mathbb{R}^n) \hookrightarrow \mathring{L}_p^r(w, \mathbb{R}^n)$ of T using (70) (and the completeness of $\mathring{L}_p^r(w, \mathbb{R}^n)$) in the same way as in the linear case. Whenever T is linear, duality directly yields the assertions

$$\tilde{T}'' : L_p^r(w, \mathbb{R}^n) \hookrightarrow L_p^r(w, \mathbb{R}^n) \quad \text{and} \quad \tilde{T}' : H^e L_{p'}(w, \mathbb{R}^n) \hookrightarrow H^e L_{p'}(w, \mathbb{R}^n).$$

Whenever T satisfies (66), then we obtain by (42) and $Tf \in L_{-\frac{n}{p},w}(\mathbb{R}^n)$ the identities

$$(71) \quad \begin{aligned} &\langle f, \tilde{T}'g \rangle_{(\mathring{L}_p^r(w, \mathbb{R}^n), H^e L_{p'}(w, \mathbb{R}^n))} = \langle \tilde{T}f, g \rangle_{(\mathring{L}_p^r(w, \mathbb{R}^n), H^e L_{p'}(w, \mathbb{R}^n))} \\ &= \langle Tf, g \rangle_{(\mathring{L}_p^r(w, \mathbb{R}^n), H^e L_{p'}(w, \mathbb{R}^n))} \stackrel{(42)}{=} \int_{\mathbb{R}^n} (Tf)(x)g(x)dx \\ &= \langle Tf, g \rangle_{(L_{p_0,w}, L_{p'_0,w'})} \stackrel{(66)}{=} \langle f, Tg \rangle_{(L_{p_0,w}, L_{p'_0,w'})} \end{aligned}$$

for all $f, g \in D(\mathbb{R}^n)$ where $w' = w^{1-p'_0}$. Therefore, $\tilde{T}'g = Tg$ almost everywhere for all $g \in D(\mathbb{R}^n)$ by the fundamental lemma of the calculus of variations. Hence, \tilde{T}' is a linear and continuous extension of T to $H^e L_{p'}(w, \mathbb{R}^n)$. Moreover, the bidual $\tilde{T}'' = (\tilde{T}')'$ is an extension of T to $L_p^r(\mathbb{R}^n)$. Indeed, we have

$$\begin{aligned} &\langle g, \tilde{T}''f \rangle_{(H^e L_{p'}(w, \mathbb{R}^n), L_p^r(w, \mathbb{R}^n))} = \langle \tilde{T}'g, f \rangle_{(H^e L_{p'}(w, \mathbb{R}^n), L_p^r(w, \mathbb{R}^n))} \\ &= \langle \tilde{T}'g, f \rangle_{(L_{u',\tilde{w}}(\mathbb{R}^n), L_{u,w}(\mathbb{R}^n))} = \int_{\mathbb{R}^n} \tilde{T}'g(x)f(x)dx \\ &\stackrel{(42)}{=} \langle f, \tilde{T}'g \rangle_{(\mathring{L}_p^r(w, \mathbb{R}^n), H^e L_{p'}(w, \mathbb{R}^n))} = \langle \tilde{T}f, g \rangle_{(\mathring{L}_p^r(w, \mathbb{R}^n), H^e L_{p'}(w, \mathbb{R}^n))} \\ &\stackrel{(71)}{=} \langle Tf, g \rangle_{(L_{p_0,w}, L_{p'_0,w'})} \end{aligned}$$

for all $f, g \in D(\mathbb{R}^n)$ where $u \equiv -\frac{n}{r}$ and $\hat{w} \equiv w^{1-u'}$. Hereby, the second equality holds by Hahn-Banach's theorem ($f \in L_p^r(w, \mathbb{R}^n)$ if, and only if, $f \in H^q L_{p'}(w, \mathbb{R}^n)'$ if, and only if, $f \in L_{u', \hat{w}}(\mathbb{R}^n)'$ if, and only if, $f \in L_{u, w}(\mathbb{R}^n)$). Therefore, $\tilde{T}'' = T$ on $D(\mathbb{R}^n)$ using (32). \square

Remark 6.2. Theorem 5.3 and Proposition 3.1 imply the inequality

$$(72) \quad \|Tf\|_{L_p^r(w, \mathbb{R}^n)} \leq c \|f\|_{L_p^r(w, \mathbb{R}^n)}$$

and $Tf \in \mathring{L}_p^r(w, \mathbb{R}^n)$ for all $f \in D(\mathbb{R}^n)$ where the constant c does not depend on f without any further assumption on T except (64). The assumptions that T is linear or that T satisfies (65) are just used to obtain (69) from (72).

6.2. Distinguished examples.

6.2.1. Calderón-Zygmund operators and its maximal truncations.

Corollary 6.3. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $-n < \varrho < -\frac{n}{p'}$ and let $w \in A_p$. Let T be an operator with domain $D(\mathbb{R}^n)$ satisfying

$$\|Tf\|_{L_2(\mathbb{R}^n)} \leq c_1 \|f\|_{L_2(\mathbb{R}^n)}$$

where the constant c_1 is independent of $f \in D(\mathbb{R}^n)$ and suppose that

$$(Tf)(y) = \lim_{\varepsilon \searrow 0} \int_{x \in \mathbb{R}^n, |y-x| \geq \varepsilon} K(y, x) f(x) dx$$

almost everywhere for all $f \in D(\mathbb{R}^n)$, where the function $K(\cdot, \cdot)$ defined $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ satisfies the conditions $|K(x, y)| \leq c_2 |x - y|^{-n}$ and

$$|K(x, y) - K(x', y)| \leq c_2 \frac{|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}},$$

$$\text{whenever } 2|x - x'| \leq \max(|x - y|, |x' - y|),$$

$$|K(x, y) - K(x, y')| \leq c_2 \frac{|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}},$$

$$\text{whenever } 2|y - y'| \leq \max(|x - y|, |x - y'|).$$

The maximal truncation of T is given by

$$(T^{(*)}f)(y) = \sup_{\varepsilon > 0} \left| \int_{x \in \mathbb{R}^n, |y-x| > \varepsilon} K(y, x) f(x) dx \right|.$$

Then the following statements hold true.

- (1) There are linear and bounded extensions of T acting in $L_p^r(w, \mathbb{R}^n)$.
- (2) There are unique continuous and bounded extensions of T and $T^{(*)}$ acting in $\mathring{L}_p^r(w, \mathbb{R}^n)$.
- (3) There is an unique linear and bounded extension of T acting in $H^q L_{p'}(w, \mathbb{R}^n)$.

Proof. The boundedness of T in $L_{p, w}(\mathbb{R}^n)$ for all $w \in A_p$ and all $1 < p < \infty$ is covered e.g. by [Gra09, Thm. 9.4.6, Cor. 9.4.7] which yields Part 1 and Part 2. The adjoint kernel of $K(x, y)$ given by $\overline{K(y, x)}$ also satisfies the required assumptions on the kernel. Hence, its corresponding operator is also bounded in $L_{p, w}(\mathbb{R}^n)$ for all $w \in A_p$ and all $1 < p < \infty$ (cf. [Gra09, Def. 8.1.2]) but its dual coincides with the operator T (with the kernel $K(x, y)$) on $D(\mathbb{R}^n)$ which implies Part 3. \square

Remark 6.4. The well-definedness of $T^{(*)}$ holds by the fact that

$$\left| \int_{x \in \mathbb{R}^n, |y-x| > \varepsilon} K(y, x) f(x) dx \right|, \quad f \in D(\mathbb{R}^n)$$

is bounded for each $\varepsilon > 0$ and $y \in \mathbb{R}^n$ as a consequence of Hölder's inequality. Hence, $(T^{(*)}f)(y)$ is well-defined for all $y \in \mathbb{R}$, but might be infinite. The boundedness of Calderón-Zygmund operators are crucial for an investigation of Navier-Stokes equations. We refer to the recent books of Triebel [Tri13, Tri14] where Navier-Stokes equations has been investigated in the context of unweighted (Morrey-type) Besov-Triebel-Lizorkin spaces. As a further key ingredient he used wavelet characterizations for the corresponding spaces (cf. [Ros12]).

6.2.2. Hörmander-Mikhlin type multipliers. We denote the Fourier transform of f on $S(\mathbb{R}^n)$ or $S'(\mathbb{R}^n)$ by \hat{f} and its inverse by \check{f} where the normalisation of \hat{f} does not matter for our purposes.

Corollary 6.5. *Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $-n < \varrho < -\frac{n}{p'}$ and let $w \in A_p$. Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ which satisfies*

$$\sup_{R>0} \left(R^{s|\alpha|-n} \sup_{R<|x|<2R} |D^\alpha m(x)| \right)^{\frac{1}{s}} < \infty \quad \text{for all } |\alpha| \leq n$$

where $1 < s \leq 2$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$ is a multi-index of non-negative integers α_j . Let T_m be the operator defined by

$$(73) \quad (T_m f)^\wedge(\xi) = m(\xi) \hat{f}(\xi), \quad f \in S(\mathbb{R}), \quad \xi \in \mathbb{R}.$$

Then the following statements hold true.

- (1) *There are linear and bounded extensions of T_m acting in $L_p^r(w, \mathbb{R}^n)$.*
- (2) *There is an unique linear and bounded extension of T_m acting in $\mathring{L}_p^r(w, \mathbb{R}^n)$.*
- (3) *There is an unique linear and bounded extension of T_m acting in $H^\varrho L_{p'}(w, \mathbb{R}^n)$.*

Proof. The boundedness of T_m in $L_{p,w}(\mathbb{R})$ for all $w \in A_p$ and all $1 < p < \infty$ is covered e.g. by [KW79, Thm. 1]. \square

6.2.3. Marcinkiewicz multipliers.

Corollary 6.6. *Let $1 < p < \infty$, $-\frac{1}{p} < r < 0$, $-1 < \varrho < -\frac{1}{p'}$ and let $w \in A_p$. Let m be a bounded function which has uniformly bounded variation on each of the dyadic sets $(-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$, $j \in \mathbb{Z}$, in \mathbb{R} . Let T_m be the operator defined analogously to (73). Then the following statements hold true.*

- (1) *There are linear and bounded extensions of T_m acting in $L_p^r(w, \mathbb{R})$.*
- (2) *There is an unique linear and bounded extension of T_m acting in $\mathring{L}_p^r(w, \mathbb{R})$.*
- (3) *There is an unique linear and bounded extension of T_m acting in $H^\varrho L_{p'}(w, \mathbb{R})$.*

Proof. The boundedness of T_m in $L_{p,w}(\mathbb{R})$ for all $w \in A_p$ and all $1 < p < \infty$ is covered e.g. by [Duo01, Thm. 8.35] and the reference given there (see also [Gra08, Thm. 5.2.2]). Moreover, we observe $T'_m = T_m(-\cdot)$ on $D(\mathbb{R}^n)$ and whenever m has

uniformly bounded variation on each of the dyadic sets $(-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$, $j \in \mathbb{Z}$, then $m(\cdot)$ has also uniformly bounded variation on these sets. \square

Remark 6.7. We want to mention that T_m is well-defined by (73) since $m\hat{f} \in L_2(\mathbb{R}^n)$ for $f \in S(\mathbb{R})$. The assumption on m is in particular satisfied if m is a bounded function which is continuously differentiable on $(-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$, $j \in \mathbb{Z}$, satisfying

$$\sup_{j \in \mathbb{Z}} \left[\int_{(-2^{j+1}, -2^j)} |m'(\xi)| d\xi + \int_{(2^j, 2^{j+1})} |m'(\xi)| d\xi \right] < \infty.$$

6.2.4. The maximal Carleson operator.

Corollary 6.8. Let $1 < p < \infty$, $-\frac{1}{p} < r < 0$ and let $w \in A_p$. Let the maximal Carleson operator be defined as

$$C_*(f)(x) \equiv \sup_{\varepsilon > 0} \sup_{\xi \in \mathbb{R}} \left| \int_{|x-y| > \varepsilon} \frac{f(y) e^{2\pi i \xi y}}{x-y} dy \right|, \quad f \in S(\mathbb{R}), \quad \xi \in \mathbb{R}.$$

Then there is an unique continuous and bounded extension of C_* acting in $\mathring{L}_p^r(w, \mathbb{R})$.

Proof. The boundedness of C_* in $L_{p,w}(\mathbb{R})$ for all $w \in A_p$ and all $1 < p < \infty$ is covered by [Gra09, Thm. 11.3.3]. \square

Remark 6.9. The well-definedness of C_* holds because of the fact that

$$\left| \int_{|x-y| > \varepsilon} \frac{f(y) e^{2\pi i \xi y}}{x-y} dy \right|$$

is bounded for each $\varepsilon > 0$ and $x \in \mathbb{R}$ as a consequence of Hölder's inequality. Hence, $C_*(f)(x)$ is well-defined for all $x \in \mathbb{R}$, but might be infinite.

6.2.5. Commutators.

Remark 6.10. The commutator of a singular integral operator T with a function b is defined as

$$[b, T](f) = bT(f) - T(bf) \quad \text{for } f \in D(\mathbb{R}^n).$$

If $1 < p < \infty$, $w \in A_p$ and $b \in BMO$, then $[b, T]$ is bounded in $L_{p,w}(\mathbb{R}^n)$. For detailed definitions and references we refer to [CMP11, page 62]. The unweighted case one finds also in [Gra09, Thm. 7.5.6]. Therefore the appropriate assertions of Corollary 6.5 holds also for these commutators.

6.2.6. Vector-valued boundednesses.

Remark 6.11. All the mentioned results hold also in the vector-valued situation since Muckenhoupt weighted boundednesses imply by extrapolation its (weighted and unweighted) vector-valued boundednesses (cf. [CMP11, page 22]). Let T_m be defined as in Corollary 6.5. Then by classical extrapolation we obtain

$$\left\| \left(\sum_{j \in \mathbb{N}} |T_m f_j|^q \right)^{\frac{1}{q}} \right\|_{L_{p,w}(\mathbb{R}^n)} \leq c \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L_{p,w}(\mathbb{R}^n)}$$

for any $1 < p, q < \infty$ and any $w \in A_p$. Hence, the assumption (62) of Theorem 5.3 is fulfilled for the following pairs of functions

$$\left(\left(\sum_{j \in \mathbb{N}} |T_m f_j|^q \right)^{\frac{1}{q}}, \left(\sum_{j \in \mathbb{N}} |f_j|^q \right)^{\frac{1}{q}} \right)$$

for all sequences $\{f_j\}$ where $f_j \in D(\mathbb{R}^n)$ and only a finite number of functions f_j are not identically zero. By (63) we deduce then

$$\left\| \left(\sum_{j \in \mathbb{N}} |T_m f_j|^q \right)^{\frac{1}{q}} \right\|_{L_p^r(w, \mathbb{R}^n)} \leq c \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L_p^r(w, \mathbb{R}^n)}$$

for any $1 < p, q < \infty$, any $-n/p < r < 0$, any $w \in A_p$ and all sequences $\{f_j\}$ where $f_j \in D(\mathbb{R}^n)$ and only a finite number of functions f_j are not identically zero.

The operator defined by $T(\{f_j\}) \equiv \left(\sum_{j \in \mathbb{N}} |T_m f_j|^q \right)^{\frac{1}{q}}$ can now be extended by the appropriate vector-valued duality assertions which are proved for the unweighted situation in [RS14] and which lead to the appropriate results of Corollary 6.5 in the unweighted vector-valued situation. In the same manner one would get these results also for the other operators where we dealt with T_m as a model case. The vector-valued assertions of these Hörmander-Mikhlin multipliers lead to the appropriate results in Morrey smoothness spaces of Besov-Triebel-Lizorkin type (c.f. [Ros12] for definitions). Taking those one can generalize different results on PDEs (Navier-Stokes equations, ...) in the recent book of Triebel [Tri14] where he just dealt with model cases relying on the vector-valued boundedness of the Riesz transform.

7. ASSOCIATED SPACES

Definition 7.1. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$.

- (1) The *associated space of $L_p^r(w, \mathbb{R}^n)$* is given by the norm

$$\left\| |f| L_p^r(w, \mathbb{R}^n)^\S \right\| \equiv \sup_g \int_{\mathbb{R}^n} |f(x)g(x)| dx$$

where the supremum is taken over all $\|g|L_p^r(w, \mathbb{R}^n)\| \leq 1$ and $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

- (2) The *associated space of $H^\varrho L_{p'}(w, \mathbb{R}^n)$* is given by the norm

$$\left\| |f| H^\varrho L_{p'}(w, \mathbb{R}^n)^\S \right\| \equiv \sup_g \int_{\mathbb{R}^n} |f(x)g(x)| dx$$

where the supremum is taken over all $\|g|H^\varrho L_{p'}(w, \mathbb{R}^n)\| \leq 1$ and $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

We recover results of [MST16].

Corollary 7.2. Let $1 < p < \infty$, $-\frac{n}{p} < r < 0$, $r + \varrho = -n$ and let $w \in A_p$. Then

$$\left\| |f| L_p^r(w, \mathbb{R}^n)^\S \right\| \cong \left\| |f| H^\varrho L_{p'}(w, \mathbb{R}^n) \right\| \quad \text{and} \quad \left\| |f| H^\varrho L_{p'}(w, \mathbb{R}^n)^\S \right\| \cong \left\| |f| L_p^r(w, \mathbb{R}^n) \right\|$$

where $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof. By (37) follows

$$\left\| |f| L_p^r(w, \mathbb{R}^n)^\sharp \right\| \leq c \|f| H^q L_{p'}(w, \mathbb{R}^n)\|, \left\| |f| H^q L_{p'}(w, \mathbb{R}^n)^\sharp \right\| \leq c \|f| L_p^r(w, \mathbb{R}^n)\|.$$

Let us assume now that $f \in L_p^r(w, \mathbb{R}^n)^\sharp$. We observe that

$$\left\| |f| L_p^r(w, \mathbb{R}^n)^\sharp \right\| = \sup_g \int_{\mathbb{R}^n} f(x)g(x)dx = \sup_g \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|$$

where the supremum is taken over all $\|g| L_p^r(w, \mathbb{R}^n)\| \leq 1$. Thus,

$$\sup_g \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \left\| |f| L_p^r(w, \mathbb{R}^n)^\sharp \right\|$$

where the supremum is taken over all $g \in D(\mathbb{R}^n)$ with $\|g| L_p^r(w, \mathbb{R}^n)\| \leq 1$. But then the left-hand side of the latter inequality coincides with the norm of a functional on $\dot{L}_p^r(w, \mathbb{R}^n)$. By (42) and (43) we achieve

$$\|f| H^q L_{p'}(w, \mathbb{R}^n)\| \leq c \sup_g \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq c \left\| |f| L_p^r(w, \mathbb{R}^n)^\sharp \right\|.$$

Using (32) and (33) we deduce analogously

$$\left\| |f| L_p^r(w, \mathbb{R}^n) \right\| \leq c \left\| |f| H^q L_{p'}(w, \mathbb{R}^n)^\sharp \right\|.$$

□

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